



Focusing of branes in warped backgrounds

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Abstract — Branes are embedded surfaces in a given background (bulk) spacetime. Assuming a warped bulk, we investigate (in analogy with the case for geodesics) the notion of *focusing* of families of such embedded, extremal 3-branes in a five dimensional background. The essential tool behind our analysis, is the well-known generalised Raychaudhuri equations for surface congruences. In particular, we find explicit solutions of these equations, which seem to show that families of 3-branes can focus along lower dimensional submanifolds depending on where the initial expansions are specified. We conclude with comments on the results obtained and possibilities about future work along similar lines.

Keywords — Raychaudhuri equations, embedded surfaces, warped spacetimes

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1. Introduction

The notion of geodesic focusing and the conditions under which it can occur, follow from the well-known analysis of the Raychaudhuri equation for the expansion θ (see [1], [2] for further details). Let us briefly recall this analysis, in order to set the background for further discussion. The Raychaudhuri equation (for timelike geodesics, assuming shear and rotation to be zero) is

$$\frac{d\theta}{d\lambda} + \frac{1}{3}\theta^2 = -R_{ab}\xi^a\xi^b, \quad (1)$$

where ξ^a is the tangent vector to the geodesic. One shows from (1) that if $\theta < 0$ for some $\lambda = \lambda_0$ and if $R_{ab}\xi^a\xi^b \geq 0$, $\theta \rightarrow -\infty$ at a *finite* value of λ (focusing). One may also convert the above equation into a second order form by using $\theta = 3(F'/F)$. In the second order form, one looks for the existence of zeros in F at finite λ , in order to establish the focusing theorem. The criterion for the existence of zeros follows from the theory of ordinary differential equations of the form $F''(x) + H(x)F(x) = 0$, and is the same as mentioned at the beginning of this paragraph. Through the use of the Einstein equation in the convergence condition, $R_{ab}\xi^a\xi^b \geq 0$ one arrives at the

so called strong energy condition and its variations (e.g. weak, null, dominant energy conditions). Physically, focusing implies the rather simple fact that in an attractive gravitational field, the worldlines of test particles tend to converge and ultimately meet. Focusing is therefore a pre-condition on the existence of spacetime singularities [3], though it can be completely benign, *i.e.* just a singularity in the congruence, without there being an actual spacetime singularity at the focal point.

Geodesics are *one dimensional* surfaces in a given background. Their generalisation to higher dimensional embedded surfaces is provided through the so-called minimal surfaces. Such surfaces arise as the solutions of the variational problem for the area action (Nambu-Goto). It is therefore quite likely that the geodesic deviation (Jacobi) equation and the Raychaudhuri equations will also have generalisations for the case of such minimal surfaces. This was achieved by Capovilla and Guven in [4]. Focusing in such a scenario (*i.e.* a congruence of surfaces as opposed to a congruence of curves [5]) is the main topic of discussion in this article. We restrict ourselves to simple examples. In particular, we assume a five dimensional warped background in which we have a four dimensional, timelike embedded surface. This is the currently fashionable braneworld picture [6]. We shall enumerate the

various criteria required to obtain results for the focusing of such a congruence of timelike hypersurfaces in a given warped background

In Section 2, we review the generalised Raychaudhuri equations for surface congruences. In Section 3, we specialise to 3-branes in a warped five dimensional background and obtain solutions of the generalised equations. Finally in Section 4, we provide some remarks and conclude with comments on future directions.

2. The generalised Raychaudhuri (Capovilla-Guven) equations

In the case of a family of geodesic curves the quantities that characterise a flow are the expansion (θ), the rotation (ω_a) and the shear (σ_a). The expansion measures the rate of change of the cross-sectional area of a bundle of geodesics. If it goes to negative infinity within a finite value of the affine parameter, we have focusing. Shearing corresponds to a change of shape (circle to ellipse, say) of the cross-sectional area and the rotation implies a twist. For families of surfaces, these quantities are replaced by a generalised expansion θ_a , a generalised shear Σ_{ab} and a generalised rotation Ω_{ab} . The index a corresponds to the coordinates of the embedded surface. Thus, we have an expansion, a rotation and a shear along each independent direction on the surface. It is obvious that these quantities are not independent – they are coupled to each other and that is what makes their evolution quite difficult to study. We shall assume shear and rotation to be zero – this is possible under certain conditions and we take it that they are satisfied. Thus, we have only one equation for the generalised expansion given by [4]

$$\nabla^a \theta_a + \frac{1}{N-D} \theta_a \theta^a + (M^2)'_i F = 0, \quad (2)$$

where N and D are the dimensions of the background and the embedded surface and

$$(M^2)'_i = K^{ab}_i K_{ab} + R_{\mu\nu\rho\sigma} E^{i\mu}_a E^{j\nu}_a E^{j\rho}_a E^{j\sigma}_a n^{ij}_i n^{ij}_i \quad (3)$$

In the above expression, E^μ_a constitutes the tangent vector basis chosen such that $g_{\mu\nu} E^{i\mu}_a E^{j\nu}_b = \eta_{ab}$ (a, b run from 1 to D). n^{ij}_i are the normals, with $g_{\mu\nu} n^{i\mu}_i n^{j\nu}_j = \delta^{ij}$ (i, j run from 1 to $N-D$). Also $g_{\mu\nu} n^{i\mu}_i E^{j\nu}_a = 0$. K^{ab}_i are the $N-D$ extrinsic curvatures (one along each normal direction) and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the background spacetime. Figure 1 provides a pictorial representation of the embedding in the $N=3, D=2$ case.

One can convert the above equation to a second order form by using $\theta_a = \partial_a \gamma$ and $\gamma = (N-D) \ln F$. This yields, finally

$$\nabla_a \nabla^a F + (M^2)'_i F = 0 \quad (4)$$

which is a variable mass wave equation on the embedded surface. Solving this with appropriate initial conditions, will therefore provide the criteria for focusing.

In an earlier paper [5], we had arrived at the criterion for focusing of strings (two dimensional timelike world sheets, 1-branes) in an arbitrary background. It turned out that, using a theorem in the theory of partial differential equations, the existence of zeros in F seem to be guaranteed if the condition

$$-{}^2R + R_{\mu\nu} E^{i\mu}_a E^{j\nu}_a > 0, \quad (5)$$

where 2R is the Ricci scalar of the worldsheet.

For the case of 3-branes in a warped background of five dimensions, $N-D=1$ and the quantity (M^2) turns out to be

$$(M^2) = -{}^4R + R_{\mu\nu} E^{i\mu}_a E^{j\nu}_a \quad (6)$$

If the worldbrane hypersurface is flat, then the quantity is further simplified and we have the equation for the expansion taking the form

$$\nabla_a \nabla^a F + R_{\mu\nu} E^{i\mu}_a E^{j\nu}_a F = 0 \quad (7)$$

For background Einstein spaces with $R_{\mu\nu} = \Lambda g_{\mu\nu}$, it is easy to see that the above equation takes the simple form

$$\nabla_a \nabla^a F + 4\Lambda F = 0 \quad (8)$$

We shall now try to analyse the abovementioned equations and look for ways to understand and quantify the notion of focusing of surfaces.

3. Explicit solutions and the meaning of focusing

Recall that the background five dimensional line element of a warped brane world model [6] is given as

$$ds^2 = e^{2\sigma} (\eta_{ab} dx^a dx^b) + d\sigma^2. \quad (9)$$

Here, σ is usually referred to as the *extra* dimension and the $\sigma = \text{constant}$ slice is a four dimensional timelike hypersurface which is the 3-brane and represents our four dimensional world.

Assuming the trivial embedding

$$t = t_1, x = x_1, y = y_1, z = z_1, \sigma = \sigma_0 \text{ (constant)} \quad (10)$$

with (t_1, x_1, y_1, z_1) being the coordinates on the 3-brane, we find that the tangents and normals take the simple forms:

$$L^{\mu} = e^{-2f(\sigma_0)}(1, 0, 0, 0, 0),$$

$$L^{\nu} = e^{-2f(\sigma_0)}(0, 1, 0, 0, 0),$$

$$L^{\lambda} = e^{-2f(\sigma_0)}(0, 0, 1, 0, 0),$$

$$L^{\mu_5} = e^{-2f(\sigma_0)}(0, 0, 0, 1, 0),$$

$$\eta^{\mu} = (0, 0, 0, 0, 1) \quad (11)$$

The induced metric on the brane is scaled Minkowski, i.e. $\eta'_{ab} = e^{-2f(\sigma_0)}\eta_{ab}$.

Using the Ricci tensor components for the background line element, it is easy to write down the generalised Raychaudhuri (Capovilla-Guven) equations. We have

$$e^{-2f(\sigma_0)} \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) F - 4f''(\sigma_0)F = 0, \quad (12)$$

where we have switched back to (t, x, y, z) as the coordinates on the 3-brane (instead of the (t, v_1, v_2, z) mentioned earlier). Normally, one does not solve a Raychaudhuri equation in order to arrive at a focusing theorem. The usual method is to look for zeros of F at finite values of the parameters (here t, x, y, z) or analyse the first order equation. However, the above equation, though a partial

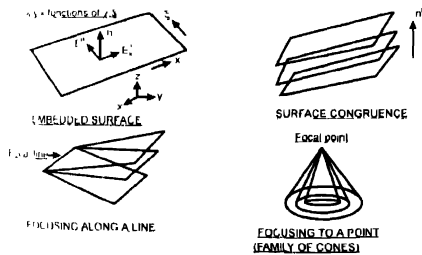


Figure 1. The top left figure shows the embedding of surfaces for the case of a two dimensional one in a 3D background. The other figures demonstrate a surface congruence, a focal curve and a focal point. The figures are all qualitative and do not pertain to any of the equations.

differential equation, is simple enough and we may solve it directly to get further insight. But, to solve the above equation, we do need boundary conditions. The central question is, where do we impose those conditions? Let us first try a simple solution of the form

$$F = A \cos(kx - \omega t + \phi_0), \quad (13)$$

where A and ϕ_0 are constants. It is easy to see that k and ω will be related via a 'dispersion relation' of the form:

$$\omega^2 = k^2 + 4f''(\sigma_0)e^{2f(\sigma_0)}. \quad (14)$$

Here, it is worth noting that the non-trivial curvature of the five dimensional background makes its appearance only through the above $\omega - k$ relation. The geometric stability criteria discussed in [7], in addition, implies $f''(\sigma_0) > 0$. The above solution for F would lead to the following expressions for the four expansions θ_i . We have

$$\theta_t = \omega \tan(kx - \omega t + \phi_0), \quad (15)$$

$$\theta_{(x,y,z)} = -k_{(x,y,z)} \tan(kx - \omega t + \phi_0) \quad (16)$$

One immediately notices a couple of important facts from the above

- (i) Assuming an initial condition defined on the 2-brane given by the equation $kx - \omega t + \phi_0 = \phi_{\text{init}}$, one can see that there is a possibility of a divergence to negative infinity (starting with a finite negative value of θ_t) in θ_t as one approaches the 2-brane surface $kx - \omega t + \phi_0 = (\pi/2)$. $\theta_{(x,y,z)}$ however, is positive initially and diverges to positive infinity at $kx - \omega t + \phi_0 = (\pi/2)$.
- (ii) Both θ_t and $\theta_{(x,y,z)}$ cannot simultaneously go to negative infinity because if one assumes a initially negative expansion for one, the other has to be positive and vice versa.

The above discussion can also be carried out assuming the solution for F as $F = A \cos(kx + \omega t + \phi_0)$. This will correspond to a time reversed situation. The general results are similar with reversal of signs at appropriate places. Alternatively, assuming a sinusoidal form or a linear combination of sines and cosines does not yield anything newer. In either case, the point to note is that if we provide initial conditions on a 2-brane surface then the focusing is on the 2-brane.

If focusing is defined to be the approach to negative infinity (at a finite value of the coordinates that define the surface) of all the expansion vector components θ_i , then we do not have focusing in this case. In other words, giving initial values on a 2-brane does not yield a focusing in the sense defined above. However, one can also argue that θ_{μ} being a vector field on the embedded surface (here, the 3-brane) the divergence of any one component would correspond to a singularity in the surface congruence. Thus, one may extend the usual notion of a focal point to include focal curves and focal surfaces. These features are illustrated in Figure 1 for the $N = 3$, $D = 2$ case.

Another worthwhile question to ask is: can we have solutions such that (i) none of the expansions diverge

anywhere, (ii) all the expansions diverge (to negative infinity) somewhere. The answer to (i) is straightforward. Assume a solution of the form

$$F = A \cos(k \cdot x - \omega t + \phi_0) \quad (17)$$

This, when substituted in the equation for F , will give a 'dispersion relation' of the form

$$\omega = -k^2 - 4f''(\sigma_0)e^{2f(\sigma_0)}. \quad (18)$$

Here, we require $f''(\sigma_0) < 0$ (which goes against the criterion of geometric stability discussed in [7]) in order to have anything meaningful. Furthermore, we need $4|f''(\sigma_0)| > k^2$. With these assumptions (if at all permissible) one can show that the expansions are all proportional to $\tanh(k \cdot x - \omega t + \phi_0)$ and, therefore, they are all finite everywhere.

The answer to the second issue (ii) is somewhat more involved. To see this, let us assume a solution in the form

$$F = T(t)X(x) \quad (19)$$

This will yield ordinary differential equations of the form

$$T + \left(\omega^2 + 4f''(\sigma_0)e^{2f(\sigma_0)} \right) T = 0, \quad (20)$$

$$(\nabla^2 + \omega^2)F = 0 \quad (21)$$

$$(22)$$

If $\omega^2 = k^2 > 0$ and $f''(\sigma_0) > 0$, then both the solutions can be oscillatory and, typically, we have

$$\theta_t = -\omega_0 \tan[\omega_0 t + \phi_0], \quad (23)$$

$$\theta_{(x,y,z)} = -k_{(x,y,z)} \tan[k \cdot x + \phi_1], \quad (24)$$

$$\omega_0 = \sqrt{4f''(\sigma_0)e^{2f(\sigma_0)} + \omega^2} \quad (25)$$

From the above, it is apparent that if we impose an initially negative expansion at some $t = t_0$ and on $k \cdot x + \phi_1 = d$ we can have all expansions going to negative infinity at finite values of t and x, y, z . The focal surface, in this case would be spacelike and two dimensional.

One can go further and assume solutions of the form

$$F = T(t)X(x)Y(y,Z) \quad (26)$$

or,

$$F = T(t)X(x)Y(y)Z(z). \quad (27)$$

In the first of these, one finds a focal curve (one dimensional) and the final one we have a focal point (zero dimensional). Let us evaluate the expansions for the latter.

These turn out to be

$$\theta_t = -\sqrt{\omega^2 + \omega_1^2} \tan\left(\sqrt{\omega^2 + \omega_1^2} t\right), \quad (28)$$

$$\theta_x = -\omega_1 \tan \omega_1 x, \quad (29)$$

$$\theta = \sqrt{\omega_1^2 + \omega_2^2} \tanh\left(\sqrt{\omega_1^2 + \omega_2^2} z\right), \quad (30)$$

$$\theta_t = -\sqrt{\omega^2 + 4f''(\sigma_0)e^{2f(\sigma_0)}} \times \tan\left(\sqrt{\omega^2 + 4f''(\sigma_0)e^{2f(\sigma_0)}} t\right), \quad (31)$$

where $\omega_{1,2}^2$ and ω^2 are appropriate separation constants. It is easy to see that in the above expressions for θ , one cannot have all of them going to negative infinity even if we assume all of them to be initially negative. In fact, one of the components (here θ_z) never diverges. Thus, providing a initially negative expansion vector field (at a given point) can lead to a focusing at a point though all the expansion coefficients will not diverge there.

Thus, the important lesson we have learnt is that the notion of focusing of surfaces is quite different (as it should be!) from the case of geodesic curves. The main difference arises because of the fact that there are several coordinates (parameters) that define a surface as opposed to the single parameter that is required to define a curve. The expansion, is thus a vector field defined on the surface and the divergence of its components at some value or set of values would indicate focusing. The notion of focusing along a special submanifold (as illustrated above) depends on where (i.e. on which submanifold) we impose the initial condition. We have also noted that all components of the vector field may not diverge to negative infinity.

In the above example, we assumed that the induced metric on the brane is flat. If this is not the case, then there are bound to be further complications. For instance, one can think of two scenarios: (a) an induced, static spherically symmetric metric on the brane, (b) an induced cosmological metric. In both these cases, the equation for the quantity F would change drastically through the wave operator defined on the curved submanifold and also through the intrinsic curvature of the induced metric. The general point made in this article about the role of where we impose initial conditions and the possibility of focusing along all possible lower dimensional submanifolds however remains unaltered, though explicit solutions will surely be very different.

Finally, it is useful to try and see what can be said

geodesic focusing by looking at the first order equation discussed at the beginning of the paper. At a general level, if one assumes $M^2 > 0$ then we have

$$\Delta \theta \leq -\frac{1}{N-D} \theta_a \theta^a \quad (32)$$

More specifically, for the warped background with a trivially embedded 3-brane (the context we have been discussing here) the first order equation gives

$$(\partial_a \theta_a + \theta_a^2) = \sum_{a=0}^4 (\partial_a \theta_a + \theta_a^2) + M^2 e^{2f(\sigma_a)} \quad (33)$$

The standard analysis for geodesic focusing can be utilised for the above equation provided we assume that the RHS of the above equation is *negative*. If such is the case, then we can say that θ_0 will go to negative infinity within a finite value of the parameter t . We cannot, however, say anything concrete about the behaviour of the other $\theta_{\alpha} = \theta(\chi, z)$ except that they should be such that the full RHS of the above equation is negative.

4 Concluding remarks

Let us now conclude with a summary of the results obtained.

- (A) In a warped background, a family of embedded branes with an initial expansion vector field, can focus along lower dimensional submanifolds (timelike or space-like) depending on where (i.e. on which submanifold) the initial expansions are specified.
- (B) (i) If the initial expansion vector components are given on a timelike 2-brane then the surface congruence will have a future expansion vector with some (but not all) of its components diverging to negative infinity. (ii) On the other hand, a specification of the initial expansion on a two dimensional section of a spacelike hypersurface can give rise to a future expansion vector with all

components diverging to negative infinity. (iii) Finally, specifying initial conditions on a curve (one dimensional) or a point (zero dimensional) will lead, once again, to a future expansion vector with some of its components diverging to negative infinity. All of the above could be termed as analogs of the usual geodesic focusing though the situation in (ii) is markedly different from the ones in (i) and (iii). One is tempted to conjecture (based on the example discussed) that for a D dimensional timelike, embedded hypersurface, all components of the expansion vector can diverge on a $D - 2$ dimensional submanifold.

- (C) As mentioned before, the above issues change quantitatively if we have curvature on the brane or we have a non-trivial embedding or if we look at Euclidean signature induced metrics. Higher codimension branes/surfaces involve further complications due to the presence of more than one normal vector. A detailed treatment of all the above cases mentioned is a topic of future investigation and will be communicated in due course.

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